


一道例题引发的“血”案
—— 向科学致敬

事情是这样的.....

- W F Chen: *Constitutive Equations for Engineering Materials*, Example 3.7, pp680
- 在 (σ, τ) 应力子空间里, Prager法则与v. Mises 准则合用时, 屈服面除了平动, 还会变形:

$$f = \frac{1}{3} \left(\sigma - \frac{2\sigma_a}{3} cd\lambda \right)^2 + (\tau - \tau_a cd\lambda)^2 - \left(k - \frac{2\sigma_a^2}{9} c^2 d\lambda^2 \right) = 0$$


然而.....


- 俺们咋也推不出wfChen的结果:

$$f = \frac{1}{3} \left(\sigma - \frac{2\sigma_a}{3} cd\lambda \right)^2 + (\tau - \tau_a cd\lambda)^2 - \left(k - \frac{2\sigma_a^2}{9} c^2 d\lambda^2 \right) = 0$$

- 我们的结果:

$$f = \frac{1}{3} (\sigma - \sigma_a cd\lambda)^2 + (\tau - \tau_a cd\lambda)^2 - k = 0$$

- 没有变形!
- 平动比wfChen的结果要快! ?




初次交锋.....

- H.B. Liu

$$f = \frac{1}{3} (\sigma - \sigma_a cd\lambda)^2 + (\tau - \tau_a cd\lambda)^2 - k = 0 \quad \checkmark$$

But $d\alpha_{ij} = c \cdot d\varepsilon_{ij}^p = c \cdot \frac{\partial f}{\partial \sigma_{ij}} d\lambda = c \cdot s_{ij} \cdot d\lambda$

$$= cd\lambda \begin{bmatrix} \frac{2}{3}\sigma_a & \tau_a & 0 \\ \tau_a & -\frac{1}{3}\sigma_a & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_a \end{bmatrix}$$


初次交锋.....


- H.B. Liu:
如果后继屈服面里出现的不是Reduced Stress里的任何一个分量, 而是别的什么形式, 那后继屈服面就不仅是初始屈服面平动的结果, 而包含了变形!

$$f = \frac{1}{3} (\sigma - \sigma_a cd\lambda)^2 + (\tau - \tau_a cd\lambda)^2 - k = 0$$

$$f = \frac{1}{3} \left(\sigma - \frac{2}{3}\sigma_a cd\lambda - \frac{1}{3}\sigma_a cd\lambda \right)^2 + (\tau - \tau_a cd\lambda)^2 - k = 0$$

$$f = \frac{1}{3} \left(\sigma - \frac{2}{3}\sigma_a cd\lambda \right)^2 + (\tau - \tau_a cd\lambda)^2 - (k - X) = 0$$

Y. L. Wong: 这是“典型忽悠”!



节外生枝.....

- 张量推导与矩阵推导的区别

-1- Matrix form in stress subspace


$$f = \frac{1}{3} \sigma^2 + \tau^2 - k = 0 \quad [d\alpha] = c[d\varepsilon^p] = \begin{Bmatrix} \frac{\partial f}{\partial \sigma} \\ \frac{\partial f}{\partial \tau} \end{Bmatrix} cd\lambda = \begin{Bmatrix} \frac{2}{3}\sigma \\ \tau \end{Bmatrix} cd\lambda$$

应使用张量应变而非工程应变

$$f = \frac{1}{3} \left(\sigma - \frac{2}{3}\sigma_a cd\lambda \right)^2 + (\tau - \tau_a cd\lambda)^2 - k = 0 \quad \frac{\partial f}{\partial \tau} = d\varepsilon_{eng12}^p = 2d\varepsilon_{12}^p$$

咋不一样哩?

-2- Tensor form

$$f = \frac{1}{3} (\sigma - \sigma_a cd\lambda)^2 + (\tau - \tau_a cd\lambda)^2 - k = 0$$


-3- Matrix form in complete stress space

$$f = \frac{1}{6} \left\{ \begin{aligned} & [(\sigma_{11} - \alpha_{11}) - (\sigma_{22} - \alpha_{22})]^2 \\ & + [(\sigma_{22} - \alpha_{22}) - (\sigma_{33} - \alpha_{33})]^2 \\ & + [(\sigma_{11} - \alpha_{11}) - (\sigma_{33} - \alpha_{33})]^2 \end{aligned} \right\} d\alpha_{ij} = cd\lambda \begin{bmatrix} \frac{2}{3}\sigma_a & \tau_a & 0 \\ \tau_a & -\frac{1}{3}\sigma_a & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_a \end{bmatrix}$$

$$+ (\sigma_{12} - \alpha_{12}) + (\sigma_{23} - \alpha_{23}) + (\sigma_{13} - \alpha_{13}) - k = 0$$

$$f = \frac{1}{3} (\sigma - \alpha_{11} + \alpha_{22})^2 + (\tau - \alpha_{12})^2 - k = 0$$

$$f = \frac{1}{3} (\sigma - \sigma_a cd\lambda)^2 + (\tau - \tau_a cd\lambda)^2 - k = 0$$

与张量形式的推导一致，且不存在保持形式的问题

再次交锋.....

• H.B. Liu:

- 首先，后继屈服面在 (σ, τ) 应力子空间里肯定变形了!
- 其次，后继屈服面在全应力空间中并没有变形，但在主应力空间中发生了转动。
- 然而，屈服面的转动是不允许的。所以wfChen说Prager法则会导致不一致的结果。
- 至于是怎么变的，我也说不清楚。

又被忽悠回来了⊗

求助祖师爷.....

- Ziegler, H., [1959], "A Modification of Prager's Hardening Rule," Quarterly of Applied Mathematics 17, pp55-65

- 使用 v. Mises 屈服准则时，无论使用Prager还是Ziegler强化法则，结果都是一样的。即无论在平面应力还是平面应变条件下，都是标准的随动强化，屈服面不会发生变形!
- 使用Tresca屈服准则时，Prager强化法则往往会致屈服面的变形!

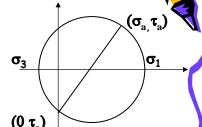
不幸的是，w.f.Chen偏偏选了v. Mises准则做例题

我们的推导是正确的⊙

进一步探究.....

• Tresca + Prager

$$\begin{bmatrix} \sigma_a & \tau_a & 0 \\ \tau_a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\sigma_1 = \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

$$\sigma_3 = \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

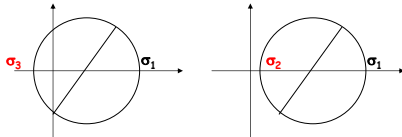
Tresca: $\left(\frac{\sigma_{11} + \sigma_{22}}{2}\right) + \sigma_{12}^2 - k = 0$

Tresca: $f = \left(\frac{\sigma_{11} + \sigma_{22}}{2}\right) + \sigma_{12}^2 - k = 0$

$$d\alpha_{ij} = c \cdot d\varepsilon_{ij}^p = c \cdot \frac{\partial f}{\partial \sigma_{ij}} d\lambda = c \cdot d\lambda \begin{bmatrix} \sigma_{11} - \sigma_{22} & \sigma_{12} & 0 \\ \sigma_{12} & -(\sigma_{11} - \sigma_{22}) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

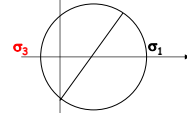
$$\bar{\sigma}_{ij} = \begin{bmatrix} \sigma - \sigma_a cd\lambda & \tau_a & 0 \\ \tau_a & \sigma_a cd\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

派生出两种可能



Case 1:

$$\bar{\sigma}_{ij} = \begin{bmatrix} \sigma - \sigma_a cd\lambda & \tau_a & 0 \\ \tau_a & \sigma_a cd\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



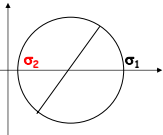
$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma - 2\sigma_a cd\lambda}{2}\right)^2 + (\tau - \tau_a cd\lambda)^2}$$

$$\sigma_3 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma - 2\sigma_a cd\lambda}{2}\right)^2 + (\tau - \tau_a cd\lambda)^2}$$

Tresca: $\bar{f} = \frac{1}{4} (\sigma - 2\sigma_a cd\lambda)^2 + (\tau - \tau_a cd\lambda)^2 - k = 0$

椭圆

Case 2:

$$\bar{\sigma}_{ij} = \begin{bmatrix} \sigma - \sigma_a cd\lambda & \tau_a & 0 \\ \tau_a & \sigma_a cd\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$


$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma - 2\sigma_a cd\lambda}{2}\right)^2 + (\tau - \tau_a cd\lambda)^2}$$

$$\sigma_3 = 0$$

Tresca: $\bar{f} = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma - 2\sigma_a cd\lambda}{2}\right)^2 + (\tau - \tau_a cd\lambda)^2} - \sqrt{k} = 0$

$$(2\sqrt{k} - \sigma_a cd\lambda) [\sigma - (\sigma_a cd\lambda + 2\sqrt{k})] = -(\tau - \tau_a cd\lambda)^2$$


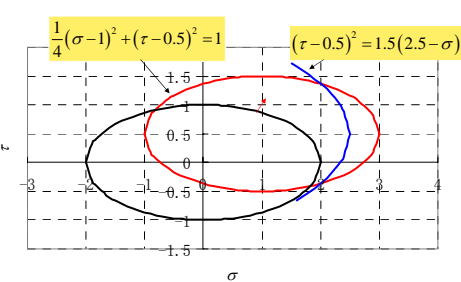
抛物线

$$\frac{1}{4}(\sigma - 2\sigma_a cd\lambda)^2 + (\tau - \tau_a cd\lambda)^2 = k$$

$$(\tau - \tau_a cd\lambda)^2 = -(2\sqrt{k} - \sigma_a cd\lambda) [\sigma - (\sigma_a cd\lambda + 2\sqrt{k})]$$

set: $\tau_a cd\lambda = 0.5$; $\sigma_a cd\lambda = 0.5$; $k = 1$, then

$$\frac{1}{4}(\sigma - 1)^2 + (\tau - 0.5)^2 = 1$$

$$(\tau - 0.5)^2 = 1.5(2.5 - \sigma)$$



Subsequent yield surface deformed in a strange way, which is not allowed!

